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On nilpotent multipliers of some verbal products of groups

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ABSTRACT

The paper is devoted to finding a homomorphic image for the c -nilpotent multiplier of the verbal product of a family of groups with respect to a variety \mathcal{V} when $\mathcal{V} \subseteq \mathcal{N}_c$ or $\mathcal{N}_c \subseteq \mathcal{V}$. Also a structure of the c -nilpotent multiplier of a special case of the verbal product, the nilpotent product, of cyclic groups is given. In fact, we present an explicit formula for the c -nilpotent multiplier of the n th nilpotent product of the group $G = \mathbf{Z}^n * \cdots * \mathbf{Z}^n * \mathbf{Z}_{r_1}^n * \cdots * \mathbf{Z}_{r_t}^n$, where r_{i+1} divides r_i for all i , $1 \leq i \leq t-1$, and $(p, r_1) = 1$ for any prime p less than or equal to $n+c$, for all positive integers n, c .

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1. Introduction and motivation

Let $G = F/R$ be a free presentation of a group G . Then the Baer invariant of G with respect to the variety \mathcal{N}_c of nilpotent groups of class at most $c \geq 1$, denoted by $\mathcal{N}_c M(G)$, is defined to be

$$\mathcal{N}_c M(G) = \frac{R \cap \gamma_{c+1}(F)}{[R, {}_c F]}.$$

$\mathcal{N}_c M(G)$ is also called the c -nilpotent multiplier of G . Clearly if $c = 1$, then $\mathcal{N}_c = \mathcal{A}$ is the variety of all abelian groups and the Baer invariant of G with respect to this variety is

$$M(G) = \frac{R \cap F'}{[R, F]},$$

which is the well-known Schur multiplier of G .

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It is important to find structures for the Schur multiplier and its generalization, the c -nilpotent multiplier, of some famous products of groups. Determining these Baer invariants of a given group is known to be very useful for the classification of groups into isoclinism classes (see [1]).

In 1907, Schur [17], using a representation method, found a structure for the Schur multiplier of a direct product of two groups. Also, Wiegold [19] obtained the same result by some properties of covering groups. In 1979 Moghaddam [13] found a formula for the c -nilpotent multiplier of a direct product of two groups, where $c + 1$ is a prime number or 4. Also, in 1998 Ellis [2] extended the formula for all $c \geq 1$. In 1997 the second author and Moghaddam [10] presented an explicit formula for the c -nilpotent multiplier of a finite abelian group for any $c \geq 1$. It is known that the direct product is a special case of the nilpotent product and we know that regular and verbal products are generalizations of the nilpotent product.

In 1972, Haebich [6] found a formula for the Schur multiplier of a regular product of a family of groups. Then the second author [9] extended the result to find a homomorphic image with a structure similar to Haebich's type for the c -nilpotent multiplier of a nilpotent product of a family of groups.

In section two, we extend the above result and find a homomorphic image for the c -nilpotent multiplier of a verbal product of a family of groups with respect to a variety \mathcal{V} when $\mathcal{V} \subseteq \mathcal{N}_c$ or $\mathcal{N}_c \subseteq \mathcal{V}$.

A special case of the verbal product of groups whose nilpotent multiplier has been studied more than others is the nilpotent product of cyclic groups. In 1992, Gupta and Moghaddam [5] calculated the c -nilpotent multiplier of the nilpotent dihedral group of class n , i.e. $G_n \cong \mathbf{Z}_2^n * \mathbf{Z}_2$. (Note that in 2001 Ellis [3] remarked that there is a slip in the statement and gave the correct one.) In 2003, Moghaddam, the second author and Kayvanfar [14] extended the previous result and calculated the c -nilpotent multiplier of the n th nilpotent product of cyclic groups for $n = 2, 3, 4$ under some conditions. Also, the second author and Parvizi [11,12] presented structures for some Baer invariants of a free nilpotent group that is the nilpotent product of infinite cyclic groups. Finally the authors and Mohammadzadeh [8] obtained an explicit formula for the c -nilpotent multiplier of the n th nilpotent product of some cyclic groups $G = \mathbf{Z}^n * \dots * \mathbf{Z}^n * \mathbf{Z}_{r_1}^n * \dots * \mathbf{Z}_{r_t}^n$, where r_{i+1} divides r_i for all i , $1 \leq i \leq t - 1$, for $c \geq n$ such that $(p, r_1) = 1$ for any prime p less than or equal to n .

In section three, we give an explicit formula for the c -nilpotent multiplier of the above group G when $(p, r_1) = 1$ for any prime p less than or equal to $n + c$, for all positive integers c, n .

2. Verbal products

A group G is said to be a *regular product* of its subgroups A_i , $i \in I$, where I is an ordered set, if the following two conditions hold:

- (i) $G = \langle A_i \mid i \in I \rangle$;
- (ii) $A_i \cap \hat{A}_i = 1$ for all $i \in I$, where $\hat{A}_i = \langle A_j \mid j \in I, j \neq i \rangle^G$.

Definition 2.1. Consider the map

$$\psi : \prod_{i \in I}^* A_i \rightarrow \prod_{i \in I}^\times A_i,$$

$$a_1 a_2 \dots a_n \mapsto (a_1, a_2, \dots, a_n),$$

which is a natural map from the free product of $\{A_i\}_{i \in I}$ on to the direct product of $\{A_i\}_{i \in I}$. Clearly its kernel is the normal closure of

$$([A_i, A_j] \mid i, j \in I, i \neq j)$$

in the free product $A = \prod_{i \in I}^* A_i$. It is denoted by $[A_i^A]$ and called *the Cartesian subgroup* of the free product (see [16] for the properties of Cartesian subgroups).

The following theorem gives a characterization of a regular product.

Theorem 2.2. (See Golovin [4].) Suppose that a group G is generated by a family $\{A_i \mid i \in I\}$ of its subgroups, where I is an ordered set. Then G is a regular product of the A_i if and only if every element of G can be written uniquely as a product

$$a_1 a_2 \dots a_n u,$$

where $1 \neq a_i \in A_{\lambda_i}$, $\lambda_1 < \dots < \lambda_n$ and $u \in [A_i^G] = \langle [A_i^G, A_j^G] \mid i, j \in I, i \neq j \rangle$.

Definition 2.3. Let \mathcal{V} be a variety of groups defined by a set of laws V . Then the verbal product of a family of groups $\{A_i\}_{i \in I}$ associated with the variety \mathcal{V} is defined to be

$$\mathcal{V} \prod_{i \in I} A_i = \frac{\prod_{i \in I}^* A_i}{V(A) \cap [A_i^A]}.$$

The verbal product is also known as varietal product or simply \mathcal{V} -product. If \mathcal{V} is the variety of all groups, then the corresponding verbal product is the free product; if $\mathcal{V} = \mathcal{A}$ is the variety of all abelian groups, then the verbal product is the direct product and if $\mathcal{V} = \mathcal{N}_c$ is the variety of all nilpotent groups of class at most c , then the verbal product will be the nilpotent product.

Let $\{A_i \mid i \in I\}$ be a family of groups and

$$1 \rightarrow R_i \rightarrow F_i \xrightarrow{\theta_i} A_i \rightarrow 1$$

be a free presentation for A_i . We denote by θ the natural homomorphism from the free product $F = \prod_{i \in I}^* F_i$ onto $A = \prod_{i \in I}^* A_i$ induced by the θ_i . Also we assume that the group G is the verbal product of $\{A_i\}_{i \in I}$ associated with the variety \mathcal{V} . If ψ is the natural homomorphism from A onto G induced by the identity map on each A_i , then we have the sequence

$$F = \prod_{i \in I}^* F_i \xrightarrow{\theta} A = \prod_{i \in I}^* A_i \xrightarrow{\psi} G = \mathcal{V} \prod_{i \in I} A_i \rightarrow 1.$$

The following notation will be used throughout this section.

Notation 2.4.

- (i) $D_1 = \prod_{i \neq j} [R_i, F_j]^F$;
- (ii) $E_c = D_1 \cap \gamma_{c+1}(F)$;
- (iii) $D_c = \prod_{\exists j \text{ s.t. } \mu_j \neq i} [R_i, F_{\mu_1}, \dots, F_{\mu_c}]^F$;
- (iv) $K_v = V(F) \cap [F_i^F]$;
- (v) $K_c = \gamma_{c+1}(F) \cap [F_i^F]$.

Let H_v be the kernel of ψ_v and R be the kernel of $\psi_v \circ \theta$. It is clear that R is actually the inverse image of H_v in F under θ , where $H_v = V(A) \cap [A_i^A]$ by the definition of the verbal product. Put $H_c = \gamma_{c+1}(A) \cap [A_i^A]$, then an immediate consequence is the following lemma.

Lemma 2.5. With the above notation we have

- (i) $\theta(K_v) = H_v$ and $\theta(K_c) = H_c$;
- (ii) $G = F/R$ and $R = \prod_{i \in I} R_i^F K_v = (\prod_{i \in I} R_i) D_1 K_v$.

Proof. (i) This follows from the definition of θ .

(ii) It is easy to see that $\ker \theta = \prod_{i \in I} R_i^F$. On the other hand, since $\theta(K_V) = \ker \psi_V$, we have $R = (\ker \theta)K_V = \prod_{i \in I} R_i^F K_V$. Also for all $r \in R_i$ and $f \in F$, $r^f = r[r, f]$. This implies that $\prod_{i \in I} R_i^F = \prod_{i \in I} R_i [R_i, F]$. Since $[R_i, F_i] \subseteq R_i$, $\prod_{i \in I} R_i^F = \prod_{i \in I} R_i D_1$. \square

We now prove some lemmas to compute the c -nilpotent multiplier of G .

Lemma 2.6. *Keeping the above notation we have*

- (i) $[R, {}^c F] = (\prod_{i \in I} [R_i, {}^c F_i]) D_c [K_V, {}^c F]$.
- (ii) If $V(F) \subseteq \gamma_{c+1}(F)$, then $R \cap \gamma_{c+1}(F) = \prod_{i \in I} (R_i \cap \gamma_{c+1}(F_i)) E_c K_V$.
- (iii) If $\gamma_{c+1}(F) \subseteq V(F)$, then $R \cap \gamma_{c+1}(F) = \prod_{i \in I} (R_i \cap \gamma_{c+1}(F_i)) E_c K_C$.

Proof. (i)

$$\begin{aligned} [R, {}^c F] &= \left[\prod_{i \in I} R_i^F K_V, {}^c F \right] \\ &= \prod_{i \in I} [R_i, {}^c F]^F [K_V, {}^c F] \\ &= \left(\prod_{i \in I} [R_i, {}^c F_i] \right) D_c [K_V, {}^c F]. \end{aligned}$$

(ii) Let $g \in R \cap \gamma_{c+1}(F)$. Then $g = r_{\lambda_1} \dots r_{\lambda_t} dk$ by Lemma 2.5, where $r_{\lambda_i} \in R_{\lambda_i}$, $d \in D_1$ and $k \in K_V$. Now consider the natural homomorphism

$$\varphi : F = \prod_{i \in I}^* F_i \rightarrow \prod_{i \in I}^\times F_i.$$

Since $g \in \gamma_{c+1}(F)$, $\varphi(g) = (r_{\lambda_1}, \dots, r_{\lambda_t}) \in \gamma_{c+1}(\prod_{i \in I}^\times F_i) = \prod_{i \in I}^\times \gamma_{c+1}(F_i)$. Therefore $r_{\lambda_i} \in \gamma_{c+1}(F_{\lambda_i}) \cap R_{\lambda_i}$ and then $dk \in \gamma_{c+1}(F) \cap [F_i^F]$. Now since $k \in V(F) \subseteq \gamma_{c+1}(F)$, we have $d \in \gamma_{c+1}(F) \cap D_1 = E_c$ and so the result follows.

(iii) Since $K_C \subseteq K_V$, $\prod_{i \in I} (R_i \cap \gamma_{c+1}(F_i)) E_c K_C \subseteq R \cap \gamma_{c+1}(F)$. For the reverse inclusion, similar to part (i), $dk \in \gamma_{c+1}(F) \cap [F_i^F]$. Therefore $R \cap \gamma_{c+1}(F) \subseteq \prod_{i \in I} (R_i \cap \gamma_{c+1}(F_i)) K_C$. Now the inclusion $E_c \subseteq K_C$ shows that the equality (iii) holds. \square

Lemma 2.7. *With the above notation, let $\varphi_c : F \rightarrow F/E_c$ be the natural homomorphism. Then $\varphi_c(\prod_{i \in I} (R_i \cap \gamma_{c+1}(F_i)) K_V)$ is the direct product of its subgroups $\varphi_c(K_V)$ and $\varphi_c(R_i \cap \gamma_{c+1}(F_i))$, $i \in I$.*

Proof. The Three Subgroups Lemma shows that

$$[R_i \cap \gamma_{c+1}(F_i), K_V] \subseteq E_c \quad \text{for all } i \in I,$$

and

$$[R_i \cap \gamma_{c+1}(F_i), R_j \cap \gamma_{c+1}(F_j)] \subseteq E_c \quad \text{for all } i, j \in I, i \neq j.$$

So we have

$$[\varphi_c(R_i \cap \gamma_{c+1}(F_i)), \varphi_c(K_V)] = 1 \quad \text{for all } i \in I,$$

and

$$[\varphi_c(R_i \cap \gamma_{c+1}(F_i)), \varphi_c(R_j \cap \gamma_{c+1}(F_j))] = 1 \quad \text{for all } i, j \in I, i \neq j.$$

Moreover, by Theorem 2.2 we conclude that

$$\varphi_c(R_i \cap \gamma_{c+1}(F_i)) \cap \left(\prod_{i \neq j} \varphi_c(R_j \cap \gamma_{c+1}(F_j)) \varphi_c(K_v) \right) = 1.$$

Now the result follows by the definition of the direct product. \square

Lemma 2.8. *With the previous notation,*

- (i) *If $V(F) \subseteq \gamma_{c+1}(F)$, then $\varphi_c(K_v)/\varphi_c([K_v, {}_cF]) \cong H_v/[H_v, {}_cA]$.*
- (ii) *If $\gamma_{c+1}(F) \subseteq V(F)$, then $\varphi_c(K_c)/\varphi_c([K_c, {}_cF]) \cong H_c/[H_v, {}_cA]$.*

Proof. (i) If $V(F) \subseteq \gamma_{c+1}(F)$, then

$$\frac{\varphi_c(K_v)}{\varphi_c([K_v, {}_cF])} \cong \frac{K_v E_c}{[K_v, {}_cF] E_c} \cong \frac{K_v}{K_v \cap [K_v, {}_cF] E_c}.$$

On the other hand

$$\frac{\theta(K_v)}{\theta([K_v, {}_cF])} \cong \frac{K_v \ker \theta}{[K_v, {}_cF] \ker \theta} \cong \frac{K_v}{K_v \cap [K_v, {}_cF] \ker \theta} \cong \frac{K_v}{K_v \cap [K_v, {}_cF] D_1 \prod_{i \in I} R_i}.$$

Now Theorem 2.2 and definition of E_c imply that

$$\frac{\theta(K_v)}{\theta([K_v, {}_cF])} \cong \frac{K_v}{K_v \cap [K_v, {}_cF] E_c}.$$

Therefore by Lemma 2.5, we conclude that

$$\frac{\varphi_c(K_v)}{\varphi_c([K_v, {}_cF])} \cong \frac{\theta(K_v)}{\theta([K_v, {}_cF])} \cong \frac{H_v}{[H_v, {}_cA]}.$$

- (ii) The proof is similar to (i). \square

Now we are ready to state and prove the main result of this section.

Theorem 2.9. *With the above notation,*

- (i) *If $\mathcal{N}_c \subseteq \mathcal{V}$, then $\prod_{i \in I}^\times \mathcal{N}_c M(A_i) \times H_v/[H_v, {}_cA]$ is a homomorphic image of $\mathcal{N}_c M(\mathcal{V} \prod_{i \in I} A_i)$, and if $\mathcal{V} \prod_{i \in I} A_i$ is finite, then the above structure is isomorphic to a subgroup of $\mathcal{N}_c M(\mathcal{V} \prod_{i \in I} A_i)$.*
- (ii) *If $\mathcal{V} \subseteq \mathcal{N}_c$, then $\prod_{i \in I}^\times \mathcal{N}_c M(A_i) \times H_c/[H_v, {}_cA]$ is a homomorphic image of $\mathcal{N}_c M(\mathcal{V} \prod_{i \in I} A_i)$, and if $\mathcal{V} \prod_{i \in I} A_i$ is finite, then the above structure is isomorphic to a subgroup of $\mathcal{N}_c M(\mathcal{V} \prod_{i \in I} A_i)$.*

Proof. (i) By Lemma 2.6(i), (ii)

$$\mathcal{N}_c M\left(\mathcal{V} \prod_{i \in I} A_i\right) \cong \frac{R \cap \gamma_{c+1}(F)}{[R, {}_cF]} \cong \frac{\prod_{i \in I} (R_i \cap \gamma_{c+1}(F_i)) E_c K_v}{\prod_{i \in I} [R_i, {}_cF_i] D_c [K_v, {}_cF]}.$$

Therefore there is a natural epimorphism from $\mathcal{N}_c M(\mathcal{V} \prod_{i \in I} A_i)$ to

$$\frac{\prod_{i \in I} (R_i \cap \gamma_{c+1}(F_i)) E_c K_v}{\prod_{i \in I} [R_i, cF_i] E_c [K_v, cF]} \cong \frac{\varphi_c(\prod_{i \in I} (R_i \cap \gamma_{c+1}(F_i)) K_v)}{\varphi_c(\prod_{i \in I} [R_i, cF_i] [K_v, cF])}.$$

Lemma 2.7 and the fact that $\varphi_c([K_v, cF]) \subseteq \varphi_c(K_v)$ and $\varphi_c([R_i, cF_i]) \subseteq \varphi_c(R_i \cap \gamma_{c+1}(F_i))$ imply that

$$\frac{\varphi_c(\prod_{i \in I} (R_i \cap \gamma_{c+1}(F_i)) K_v)}{\varphi_c(\prod_{i \in I} [R_i, cF_i] [K_v, cF])} \cong \prod_{i \in I}^{\times} \frac{\varphi_c(R_i \cap \gamma_{c+1}(F_i))}{\varphi_c([R_i, cF_i])} \times \frac{\varphi_c(K_v)}{\varphi_c([K_v, cF])}.$$

It is straightforward to see that

$$\frac{\varphi_c(R_i \cap \gamma_{c+1}(F_i))}{\varphi_c([R_i, cF_i])} \cong \frac{R_i \cap \gamma_{c+1}(F_i)}{[R_i, cF_i]}$$

by Theorem 2.2. Therefore, the result holds by Lemma 2.8(i).

(ii) By an argument similar to (i), we obtain the result. \square

We need the following lemma whose proof is straightforward.

Lemma 2.10. Let $\{A_i \mid i \in I\}$ be a family of groups. Put $A = \prod_{i \in I}^* A_i$. Then for all integers $m \geq 2$,

$$\gamma_m(A) = \prod_{i \in I} \gamma_m(A_i) (\gamma_m(A) \cap [A_i^A]).$$

In particular if the A_i are cyclic, then $\gamma_m(A) = \gamma_m(A) \cap [A_i^A]$.

The following corollary is an interesting consequence of Theorem 2.9 for cyclic groups.

Corollary 2.11. Let $\{A_i \mid i \in I\}$ be a family of cyclic groups. Then

- (i) If $\mathcal{N}_c \subseteq \mathcal{V}$, then $\mathcal{N}_c M(\mathcal{V} \prod_{i \in I} A_i) \cong H_v/[H_v, cA]$. Moreover if $\mathcal{V} \subseteq \mathcal{N}_{2c}$, then $V(\prod_{i \in I}^{2c} A_i)$ is a homomorphic image of $\mathcal{N}_c M(\mathcal{V} \prod_{i \in I} A_i)$.
- (ii) If $\mathcal{V} \subseteq \mathcal{N}_c$, then $\mathcal{N}_c M(\mathcal{V} \prod_{i \in I} A_i) \cong H_c/[H_v, cA]$. Moreover if $\mathcal{N}_m \subseteq \mathcal{V}$, then $\gamma_{c+1}(\prod_{i \in I}^{m+c} A_i)$ is a homomorphic image of $\mathcal{N}_c M(\mathcal{V} \prod_{i \in I} A_i)$.

Proof. (i) Since the A_i are cyclic groups and the R_i have no commutators, it is concluded that $D_c = E_c$. So the epimorphism in the proof of Theorem 2.9, is actually an isomorphism. Also $\mathcal{N}_c M(A_i) = 1$, therefore $\mathcal{N}_c M(\mathcal{V} \prod_{i \in I} A_i) \cong H_v/[H_v, cA]$. Now suppose $\mathcal{N}_c \subseteq \mathcal{V} \subseteq \mathcal{N}_{2c}$. The inclusion $V(A) \subseteq \gamma_{c+1}(A)$ and Lemma 2.10 imply that $V(A) \subseteq [A_i^A]$ and thus $H_v = V(A) \cap [A_i^A] = V(A)$. So we have $\mathcal{N}_c M(\mathcal{V} \prod_{i \in I} A_i) = V(A)/[V(A), cA]$ and hence $V(A)/\gamma_{2c+1}(A)$ is a homomorphic image of $\mathcal{N}_c M(\mathcal{V} \prod_{i \in I} A_i)$. On the other hand since $\mathcal{V} \subseteq \mathcal{N}_{2c}$, we have $V(A)/\gamma_{2c+1}(A) = V(A/\gamma_{2c+1}(A)) = V(\prod_{i \in I}^{2c} A_i)$. This completes the proof.

(ii) An argument similar to (i), shows that $\mathcal{N}_c M(\mathcal{V} \prod_{i \in I} A_i) \cong H_c/[H_v, cA]$. Now since $\mathcal{N}_m \subseteq \mathcal{V} \subseteq \mathcal{N}_c$, $\gamma_{c+1}(A)/\gamma_{m+c+1}(A)$ is a homomorphic image of $\mathcal{N}_c M(\mathcal{V} \prod_{i \in I} A_i)$ and also

$$\frac{\gamma_{c+1}(A)}{\gamma_{m+c+1}(A)} = \gamma_{c+1} \left(\frac{A}{\gamma_{m+c+1}(A)} \right) = \gamma_{c+1} \left(\prod_{i \in I}^{m+c} A_i \right).$$

Hence the result follows. \square

Remark 2.12. Let $\{A_i \mid i \in I\}$ be a family of groups.

- (i) If \mathcal{V} is the variety of trivial groups, then Theorem 2.9 implies that $\prod_{i \in I}^{\times} \mathcal{N}_c M(A_i)$ is a homomorphic image of $\mathcal{N}_c M(\prod_{i \in I}^* A_i)$. In particular $M(\prod_{i \in I}^* A_i) = \prod_{i \in I}^{\times} M(A_i)$ which is a result of Miller [15].
- (ii) If \mathcal{V} is the variety of nilpotent groups of class at most n , \mathcal{N}_n , then main results of the second author [9] are obtained by Theorem 2.9 and Corollary 2.11.

3. Nilpotent products of cyclic groups

In this section we use a result of the previous section and find a structure for the c -nilpotent multiplier of the group $G = \mathbf{Z}^n * \cdots * \mathbf{Z}^n * \mathbf{Z}_{r_1}^n * \cdots * \mathbf{Z}_{r_t}^n$, where r_{i+1} divides r_i for all i , $1 \leq i \leq t-1$, such that $(p, r_1) = 1$ for any prime p less than or equal to $n+c$. The proof relies on basic commutators [7] and related results. We recall that the number of basic commutators of weight c on n generators, denoted by $\chi_c(n)$, is determined by Witt formula [7]. Also, M. Hall proved that if F is the free group on free generators x_1, x_2, \dots, x_r and c_1, \dots, c_t are basic commutators of weight $1, 2, \dots, n$, on x_1, \dots, x_r , then an arbitrary element f of F has a unique representation,

$$f = c_1^{\beta_1} c_2^{\beta_2} \cdots c_t^{\beta_t} \mod \gamma_{n+1}(F).$$

In particular the basic commutators of weight n provide a basis for the free abelian group $\gamma_n(F)/\gamma_{n+1}(F)$ (see [7]).

The following theorem represents the elements of some nilpotent products of cyclic groups in terms of basic commutators.

Theorem 3.1. (See [18].) Let A_1, \dots, A_t be cyclic groups of order $\alpha_1, \dots, \alpha_t$ respectively, where if A_i is infinite cyclic, then $\alpha_i = 0$. Let a_i generate A_i and let $G = A_1 * \cdots * A_t$, where n is greater than or equal to 2. Suppose that all the primes appearing in the factorizations of the α_i are greater than or equal to n and u_1, u_2, \dots , are basic commutators of weight less than n , on the letters a_1, \dots, a_t . Put $N_i = \alpha_{i_j}$ if $u_i = a_{i_j}$ of weight 1, and

$$N_i = \gcd(\alpha_{i_1}, \dots, \alpha_{i_k})$$

if a_{i_j} , $1 \leq j \leq k$, appears in u_i . Then every element g of G can be uniquely expressed as

$$g = \prod u_i^{m_i},$$

where the m_i are integers modulo N_i (by gcd we mean the greatest common divisor).

The following theorem is an interesting consequence of Corollary 2.11.

Theorem 3.2. Let $\{A_i \mid i \in I\}$ be a family of cyclic groups. Then

- (i) if $n \geq c$, then $\mathcal{N}_c M(\prod_{i \in I}^* A_i) \cong \gamma_{n+1}(\prod_{i \in I}^{n+c} A_i)$;
- (ii) if $c \geq n$, then $\mathcal{N}_c M(\prod_{i \in I}^* A_i) \cong \gamma_{c+1}(\prod_{i \in I}^{n+c} A_i)$.

Proof. (i) Put $\mathcal{V} = \mathcal{N}_n$ in Corollary 2.11 and deduce that

$$\mathcal{N}_c M\left(\prod_{i \in I}^* A_i\right) \cong H_n/[H_n, {}_c A].$$

On the other hand by Lemma 2.10, $H_n = \gamma_{n+1}(A) \cap [A_i^A] = \gamma_{n+1}(A)$. Therefore

$$\mathcal{N}_c M \left(\prod_{i \in I}^n A_i \right) \cong \frac{\gamma_{n+1}(A)}{[\gamma_{n+1}(A), cA]} = \gamma_{n+1} \left(\frac{A}{\gamma_{n+c+1}(A)} \right) = \gamma_{n+1} \left(\prod_{i \in I}^{n+c} A_i \right).$$

(ii) The result follows as for (i). \square

Now, we are in a position to state and prove the main result of this section.

Theorem 3.3. Let $G = A_1 \overset{n}{*} \dots \overset{n}{*} A_{m+t}$ be the n th nilpotent product of cyclic groups such that $A_i \cong \mathbf{Z}$ for $1 \leq i \leq m$ and $A_{m+j} \cong \mathbf{Z}_{r_j}$ and $r_{j+1} \mid r_j$ for all $1 \leq j \leq t-1$. If $(p, r_1) = 1$ for any prime p less than or equal to $n+c$, then

- (i) if $n \geq c$, then $\mathcal{N}_c M(G) \cong \mathbf{Z}^{(g_0)} \oplus \mathbf{Z}_{r_1}^{(g_1 - g_0)} \oplus \dots \oplus \mathbf{Z}_{r_t}^{(g_t - g_{t-1})}$;
(ii) if $c \geq n$, then $\mathcal{N}_c M(G) \cong \mathbf{Z}^{(f_0)} \oplus \mathbf{Z}_{r_1}^{(f_1 - f_0)} \oplus \dots \oplus \mathbf{Z}_{r_t}^{(f_t - f_{t-1})}$,

where $f_k = \sum_{i=1}^n \chi_{c+i}(m+k)$ and $g_k = \sum_{i=1}^c \chi_{n+i}(m+k)$ for $0 \leq k \leq t$ and $\mathbf{Z}_r^{(d)}$ denotes the direct sum of d copies of the cyclic group \mathbf{Z}_r .

Proof. (i) If $n \geq c$, then by Theorem 3.2, it is enough to find the structure of $\gamma_{n+1}(\prod_{i \in I}^{n+c} A_i)$. Suppose that $A_i = \langle a_i \mid a_i^{\alpha_i} \rangle$ for $1 \leq i \leq m+t$ such that $\alpha_i = 0$ for $1 \leq i \leq m$ and $\alpha_{m+i} = r_i$ for $1 \leq i \leq t$. Also let F be the free group generated by a_1, \dots, a_{m+t} and B be the set of all basic commutators of weight $1, 2, \dots, c+n$ on the letters a_1, \dots, a_{m+t} . Now define

$$D = \{u^{N_i} \mid u \in B \text{ and } N_i = \gcd(\alpha_{i_1}, \dots, \alpha_{i_k}) \text{ if } a_{i_j} \text{ appears in } u \text{ for } 1 \leq j \leq k\}.$$

Then Theorem 3.1 implies that $\prod_{i \in I}^{n+c} A_i = F / \langle D \rangle \gamma_{c+n+1}(F)$ and so

$$\begin{aligned} \gamma_{n+1} \left(\prod_{i \in I}^{n+c} A_i \right) &= \gamma_{n+1} \left(\frac{F}{\langle D \rangle \gamma_{c+n+1}(F)} \right) \\ &= \frac{\gamma_{n+1}(F)}{\langle D \rangle \gamma_{c+n+1}(F) \cap \gamma_{n+1}(F)} \\ &\cong \frac{\gamma_{n+1}(F) / \gamma_{c+n+1}(F)}{(\langle D \rangle \cap \gamma_{n+1}(F)) \gamma_{c+n+1}(F) / \gamma_{c+n+1}(F)}. \end{aligned}$$

It can be deduced from Hall Theorem that $\gamma_{n+1}(F) / \gamma_{c+n+1}(F)$ is a free abelian group with a basis $\bar{B}_1 = \{u \gamma_{c+n+1}(F) \mid u \in B_1\}$, where B_1 is the set of all basic commutators of weight $n+1, \dots, c+n$ on a_1, \dots, a_{m+t} . Also, the uniqueness of the presentation of elements implies that the abelian group $(\langle D \rangle \cap \gamma_{n+1}(F)) \gamma_{c+n+1}(F) / \gamma_{c+n+1}(F)$ is free with a basis

$$\bar{E} = \left\{ u \gamma_{c+n+1}(F) \mid u \in \bigcup_{j=1}^t D_j \right\},$$

where D_j is the set of all u^{r_j} , such that u is a basic commutator of weight $n+1, \dots, c+n$ on a_1, \dots, a_{m+j} such that a_{m+j} appears in u . Also we have

$$|D_j| = \sum_{i=1}^c \chi_{n+i}(m+j) - \chi_{n+i}(m+j-1) = g_j - g_{j-1}.$$

This completes the proof.

(ii) The proof is similar to (i). \square

Note that the authors with F. Mohammadzadeh [8] by a different method presented a similar structure for $\mathcal{N}_c M(G)$, for $c \geq n$ with a weaker condition $(p, r_1) = 1$ for any prime p less than or equal to n .

Remark 3.4. The condition $r_{j+1} \mid r_j$, in the above theorem, simplifies the structure of the c -nilpotent multiplier of G and gives a clear formula. One can use the above method and find the structure of $\mathcal{N}_c M(G)$ without the condition $r_{j+1} \mid r_j$, but with a more complex formula. For example, for a simple case if $G = \mathbf{Z}_r * \mathbf{Z}_s^n$ where $(p, r) = (p, s) = 1$ for any prime p less than or equal to $n+c$ and $(r, s) = d$, then

- (i) if $n \geq c$, then $\mathcal{N}_c M(G) \cong \mathbf{Z}_d^{(\sum_{i=1}^c \chi_{n+i}(2))}$;
- (ii) if $c \geq n$, then $\mathcal{N}_c M(G) \cong \mathbf{Z}_d^{(\sum_{i=1}^n \chi_{c+i}(2))}$.

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